A New Generalized Kumaraswamy Distribution

Jalmar M. F. Carrasco *
Departamento de Estatística
Universidade de São Paulo
Brazil

Silvia L. P. Ferrari [†] Departamento de Estatística Universidade de São Paulo Brazil

Gauss M. Cordeiro. [‡]
Departamento de Estatística e Informática
Universidade Federal Rural de Pernambuco
Brazil

Abstract

A new five-parameter continuous distribution which generalizes the Kumaraswamy and the beta distributions as well as some other well-known distributions is proposed and studied. The model has as special cases new four- and three-parameter distributions on the standard unit interval. Moments, mean deviations, Rényi's entropy and the moments of order statistics are obtained for the new generalized Kumaraswamy distribution. The score function is given and estimation is performed by maximum likelihood. Hypothesis testing is also discussed. A data set is used to illustrate an application of the proposed distribution.

Keywords: Beta distribution; Continuous proportions; Generalized Kumaraswamy distribution; Kumaraswamy distribution; Maximum likelihood; McDonald Distribution; Moments.

1 Introduction

We introduce a new five-parameter distribution, so-called generalized Kumaraswamy (GKw) distribution, which contains some well-known distributions as special sub-models as, for example, the Kumaraswamy (Kw) and beta (\mathfrak{B}) distributions. The GKw distribution allows us to define new three- and four-parameter generalizations of such distributions. The new model can be used in a variety of problems for modeling continuous proportions data due to its flexibility in accommodating different forms of density functions.

^{*}E-mail: jalmar@ime.usp.br

[†]Corresponding author: Departamento de Estatística, Universidade de São Paulo, Rua do Matão, 1010, 05508-090, São Paulo, SP, Brazil. E-mail: silviaferrari.usp@gmail.com

[‡]E-mail: gausscordeiro@uol.com.br

The GKw distribution comes from the following idea. Wahed (2006) and Ferreira and Steel (2006) demonstrated that any parametric family of distributions can be incorporated into larger families through an application of the probability integral transform. Specifically, let $G_1(\cdot; \boldsymbol{\omega})$ be a cumulative distribution function (cdf) with corresponding probability density function (pdf) $g_1(\cdot; \boldsymbol{\omega})$, and $g_2(\cdot; \boldsymbol{\tau})$ be a pdf having support on the standard unit interval. Here, $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ represent scalar or vector parameters. Now let

$$F(x; \boldsymbol{\omega}, \boldsymbol{\tau}) = \int_0^{G_1(x; \boldsymbol{\omega})} g_2(t; \boldsymbol{\tau}) dt.$$
 (1)

Note that $F(\cdot; \boldsymbol{\omega}, \boldsymbol{\tau})$ is a cdf and that $F(\cdot; \boldsymbol{\omega}, \boldsymbol{\tau})$ and $G_1(x; \boldsymbol{\omega})$ have the same support. The pdf corresponding to (1) is

$$f(x; \boldsymbol{\omega}, \boldsymbol{\tau}) = g_2(G_1(x; \boldsymbol{\omega}); \boldsymbol{\tau})g_1(x; \boldsymbol{\omega}). \tag{2}$$

This mechanism for defining generalized distributions from a parametric cdf $G_1(\cdot; \omega)$ is particularly attractive when $G_1(\cdot; \omega)$ has a closed-form expression.

The beta density is often used in place of $g_2(\cdot; \tau)$. However, different choices for $G_1(\cdot; \omega)$ have been considered in the literature. Eugene et al. (2002) defined the beta normal distribution by taking $G_1(\cdot;\omega)$ to be the cdf of the standard normal distribution and derived some of its first moments. More general expressions for these moments were obtained by Gupta and Nadarajah (2004a). Nadarajah and Kotz (2004) introduced the beta Gumbel (BG) distribution by taking $G_1(\cdot;\omega)$ to be the cdf of the Gumbel distribution and provided closed form expressions for the moments, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation procedure. Nadarajah and Gupta (2004) introduced the beta Fréchet (BF) distribution by taking $G_1(\cdot;\omega)$ to be the Fréchet distribution, derived the analytical shapes of its density and hazard rate functions and calculated the asymptotic distribution of its extreme order statistics. Also, Nadarajah and Kotz (2006) dealt with the beta exponential (BE) distribution and obtained its moment generating function, its first four cumulants, the asymptotic distribution of its extreme order statistics and discussed maximum likelihood estimation.

The starting point of our proposal is the Kumaraswamy (Kw) distribution (Kumaraswamy, 1980; see also Jones, 2009). It is very similar to the beta distribution but has a closed-form cdf given by

$$G_1(x; \boldsymbol{\omega}) = 1 - (1 - x^{\alpha})^{\beta}, \ 0 < x < 1,$$
 (3)

where $\boldsymbol{\omega} = (\alpha, \beta)^{\top}$, $\alpha > 0$ and $\beta > 0$. Its pdf becomes

$$g_1(x; \boldsymbol{\omega}) = \alpha \beta x^{\alpha - 1} (1 - x^{\alpha})^{\beta - 1}, \ 0 < x < 1.$$
 (4)

If X is a random variable with pdf (4), we write $X \sim \text{Kw}(\alpha, \beta)$. The Kw distribution was originally conceived to model hydrological phenomena and has been used for this and also for

other purposes. See, for example, Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi et al. (2000), Ganji et al. (2006), Sanchez et al. (2007) and Courard-Hauri (2007).

In the present paper, we propose a generalization of the Kw distribution by taking $G_1(\cdot; \boldsymbol{\omega})$ as cdf (3) and $g_2(\cdot; \boldsymbol{\tau})$ as the standard generalized beta density of first kind (McDonald, 1984), with pdf given by

$$g_2(x; \boldsymbol{\tau}) = \frac{\lambda x^{\lambda \gamma - 1} (1 - x^{\lambda})^{\eta - 1}}{B(\gamma, \eta)}, \quad 0 < x < 1, \tag{5}$$

where $\boldsymbol{\tau} = (\gamma, \eta, \lambda)^{\top}$, $\gamma > 0, \eta > 0$ and $\lambda > 0$, $B(\gamma, \eta) = \Gamma(\gamma)\Gamma(\eta)/\Gamma(\gamma + \eta)$ is the beta function and $\Gamma(\cdot)$ is the gamma function. If X is a random variable with density function (5), we write $X \sim \text{GB1}(\gamma, \eta, \lambda)$. Note that if $X \sim \text{GB1}(\gamma, \eta, 1)$ then $X \sim \mathfrak{B}(\gamma, \eta)$, i.e., X has a beta distribution with parameters γ and η .

The article is organized as follows. In Section 2, we define the GKw distribution, plot its density function for selected parameter values and provide some of its mathematical properties. In Section 3, we present some special sub-models. In Section 4, we obtain expansions for the distribution and density functions. We demonstrate that the GKw density can be expressed as a mixture of Kw and power densities. In Section 5, we give general formulae for the moments and the moment generating function. Section 6 provides an expansion for the quantile function. Section 7 is devoted to mean deviations about the mean and the median and Bonferroni and Lorenz curves. In Section 8, we derive the density function of the order statistics and their moments. The Rényi entropy is calculated in Section 9. In Section 10, we discuss maximum likelihood estimation and determine the elements of the observed information matrix. Section 11 provides an application to a real data set. Section 12 ends the paper with some conclusions.

2 The New Distribution

We obtain an appropriate generalization of the Kw distribution by taking $G_1(\cdot; \omega)$ as the two-parameter Kw cdf (3) and associated pdf (4). For $g_2(\cdot; \tau)$, we consider a three-parameter generalized beta density of first kind given by (5). To avoid non-identifiability problems, we allow η to vary on $[1, \infty)$ only. We then write $\delta = \eta - 1$ which varies on $(0, \infty)$. Using (1), the cdf of the GKw distribution, with five positive parameters α , β , γ , δ and λ , is defined by

$$F(x;\boldsymbol{\theta}) = \frac{\lambda}{B(\gamma,\delta+1)} \int_0^{1-(1-x^{\alpha})^{\beta}} y^{\gamma\lambda-1} (1-y^{\lambda})^{\delta} dy, \tag{6}$$

where $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \delta, \lambda)^{\top}$ is the parameter vector.

The pdf corresponding to (6) is straightforwardly obtained from (2) as

$$f(x; \boldsymbol{\theta}) = \frac{\lambda \alpha \beta x^{\alpha - 1}}{B(\gamma, \delta + 1)} (1 - x^{\alpha})^{\beta - 1} [1 - (1 - x^{\alpha})^{\beta}]^{\gamma \lambda - 1} \{1 - [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}\}^{\delta}, \ 0 < x < 1.$$
 (7)

Based on the above construction, the new distribution can also be referred to as the McDonald Kumaraswamy (McKw) distribution. If X is a random variable with density function (7), we write $X \sim GKw(\alpha, \beta, \gamma, \delta, \lambda)$.

An alternative, but related, motivation for (6) comes through the beta construction (Eugene et al., 2002). We can easily show that

$$F(x; \boldsymbol{\theta}) = I_{[1-(1-x^{\alpha})^{\beta}]^{\lambda}}(\gamma, \delta + 1), \tag{8}$$

where $I_x(a,b) = B(a,b)^{-1} \int_0^x \omega^{a-1} (1-\omega)^{b-1} d\omega$ denotes the incomplete beta function ratio. Thus, the GKw distribution can arise by taking the beta construction applied to a new distribution, namely the exponentiated Kumaraswamy (EKw) distribution, to yield (7), which can also be called the beta exponentiated Kumaraswamy (BEKw) distribution, i.e., a beta type distribution defined by the baseline cumulative function $G(x) = [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}$.

Immediately, inverting the transformation motivation (8), we can generate X following the GKw distribution by $X = [1 - (1 - V^{1/\lambda})^{1/\beta}]^{1/\alpha}$, where V is a beta random variable with parameters γ and $\delta + 1$. This scheme is useful because of the existence of fast generators for beta random variables. Figure 1 plots some of the possible shapes of the density function (7). The GKw density function can take various forms, bathtub, J, inverted J, monotonically increasing or decreasing and upside-down bathtub, depending on the parameter values.

We now provide two properties of the GKw distribution.

Proposition 1. If $X \sim GKw(1, \beta, \gamma, \delta, \lambda)$, then $Y = X^{1/\alpha} \sim GKw(\alpha, \beta, \gamma, \delta, \lambda)$ for $\alpha > 0$.

Proposition 2. Let $X \sim GKw(\alpha, \beta, \gamma, \delta, \lambda)$ and $Y = -\log(X)$. Then, the pdf of Y is given by

$$f(y; \boldsymbol{\theta}) = \frac{\lambda \alpha \beta}{B(\gamma, \delta + 1)} e^{-\alpha y} (1 - e^{-\alpha y})^{\beta - 1} [1 - (1 - e^{-\alpha y})^{\beta}]^{\gamma \lambda - 1} \{1 - [1 - (1 - e^{-\alpha y})^{\beta}]^{\lambda}\}^{\delta}, \ y > 0. \ (9)$$

We call (9) the log-generalized Kumaraswamy (LGKw) distribution.

3 Special Sub-Models

The GKw distribution is very flexible and has the following distributions as special submodels.

The Kumaraswamy distribution (Kw)

If $\lambda = \gamma = 1$ and $\delta = 0$, the GKw distribution reduces to the Kw distribution with parameters α and β , and cdf and pdf given by (3) and (4), respectively.

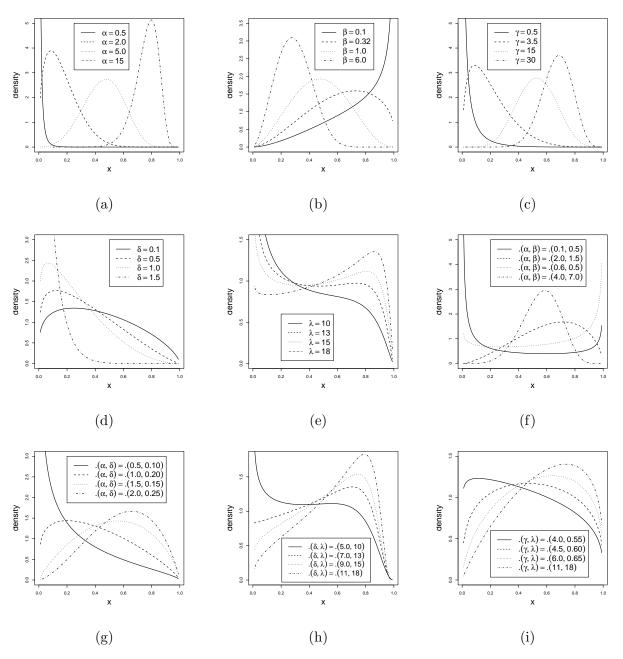


Figure 1: GKw density curves. (a) $\boldsymbol{\theta} = (\alpha, 3.5, 1.5, 2.5, 0.5)^{\top}$, (b) $\boldsymbol{\theta} = (3.5, \beta, 1.5, 2.5, 0.5)^{\top}$, (c) $\boldsymbol{\theta} = (1.0, 1.5, \gamma, 2.5, 0.5)^{\top}$, (d) $\boldsymbol{\theta} = (1.0, 1.5, 2.5, \delta, 0.5)^{\top}$, (e) $\boldsymbol{\theta} = (0.5, 0.7, 0.1, 3.0, \lambda)^{\top}$, (f) $\boldsymbol{\theta} = (\alpha, \beta, 2.5, 0.1, 0.5)^{\top}$, (g) $\boldsymbol{\theta} = (\alpha, 1.5, 2.5, \delta, 0.5)^{\top}$, (h) $\boldsymbol{\theta} = (0.5, 0.7, 0.15, \delta, \lambda)^{\top}$, (i) $\boldsymbol{\theta} = (0.5, 1.0, \gamma, 0.3, \lambda)^{\top}$.

The McDonald distribution (Mc)

For $\alpha = \beta = 1$, we obtain the Mc distribution (5) with parameters γ , $\delta + 1$ and λ .

The beta distribution

If $\alpha = \beta = \lambda = 1$, the GKw distribution reduces to the beta distribution with parameters γ and $\delta + 1$.

The beta Kumaraswamy distribution (BKw)

If $\lambda = 1$, (7) yields

$$f(x; \alpha, \beta, \gamma, \delta, 1) = \frac{\alpha \beta x^{\alpha - 1}}{B(\gamma, \delta + 1)} (1 - x^{\alpha})^{\beta(\delta + 1) - 1} [1 - (1 - x^{\alpha})^{\beta}]^{\gamma - 1}, \ 0 < x < 1.$$

This distribution can be viewed as a four-parameter generalization of the Kw distribution. We refer to it as the BKw distribution since its pdf can be obtained from (2) by setting $G_1(x; \omega)$ to be the $Kw(\alpha, \beta)$ cdf and $g_2(x; \tau)$ to the $\mathfrak{B}(\gamma, \delta + 1)$ density function.

The Kumaraswamy-Kumaraswamy distribution (KwKw)

For $\gamma = 1$, (7) reduces to (for 0 < x < 1)

$$f(x; \alpha, \beta, \gamma, \delta, \lambda) = \lambda \alpha \beta (\delta + 1) x^{\alpha - 1} (1 - x^{\alpha})^{\beta - 1} [1 - (1 - x^{\alpha})^{\beta}]^{\lambda - 1} \{1 - [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}\}^{\delta}.$$

Again, this distribution is a four-parameter generalization of the Kw distribution. It can be obtained from (2) by replacing $G_1(x; \omega)$ by the cdf of the $Kw(\alpha, \beta)$ distribution and $g_2(x; \tau)$ by the pdf of the $Kw(\gamma, \delta + 1)$ distribution. Its cdf has a closed form given by

$$F(x; \alpha, \beta, 1, \delta, \lambda) = 1 - \{1 - [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}\}^{\delta + 1}.$$

The EKw distribution

If $\delta = 0$ and $\gamma = 1$, (7) gives

$$f(x; \alpha, \beta, 1, 0, \lambda) = \lambda \alpha \beta x^{\alpha - 1} (1 - x^{\alpha})^{\beta - 1} [1 - (1 - x^{\alpha})^{\beta}]^{\lambda - 1}, \ 0 < x < 1.$$

It can be easily seen that the associated cdf can be written as

$$F(x; \alpha, \beta, 1, 0, \lambda) = G_1(x; \alpha, \beta)^{\lambda},$$

where $G_1(x; \alpha, \beta)$ is the cdf of the $Kw(\alpha, \beta)$ distribution. This distribution was defined before as the EKw distribution which is a new three-parameter generalization of the Kw distribution.

The beta power distribution (BP)

For $\alpha = 1$ and $\beta = 1$, (9) reduces to

$$f(x; 1, 1, \gamma, \delta, \lambda) = \frac{\lambda}{B(\gamma, \delta + 1)} x^{\gamma \lambda - 1} (1 - x^{\lambda})^{\delta}, \quad 0 < x < 1.$$

This density function can be obtained from (2) if $G_1(x) = x^{\lambda}$ and $g_2(x)$ is taken as the beta density with parameters γ and $\delta + 1$. We call this distribution as the BP distribution.

The LGKw distribution (9) contains as special sub-models the following distributions.

The beta generalized exponential distribution (BGE)

For $\lambda = 1$, (9) gives

$$f(y; \alpha, 1, \gamma, \delta, \lambda) = \frac{\alpha\beta}{B(\gamma, \delta + 1)} e^{-\alpha y} (1 - e^{-\alpha y})^{\beta(\delta + 1) - 1} [1 - (1 - e^{-\alpha y})^{\beta}]^{\gamma - 1}, \ y > 0,$$
(10)

which is the density function of the *BGE* distribution introduced by Barreto-Souza et al. (2010). If $\gamma = 1$ and $\delta = 0$ in addition to $\lambda = 1$, the *LGKw* distribution becomes the generalized exponential distribution (Gupta and Kundu, 1999). If $\lambda = \beta = \gamma = 1$ and $\delta = 0$, (10) coincides with the exponential distribution with mean $1/\alpha$.

The beta exponential distribution (BE)

For $\beta = 1$ and $\lambda = 1$, (9) reduces to

$$f(y; \alpha, 1, \gamma, \delta, 1) = \frac{\alpha}{B(\gamma, \delta + 1)} e^{-\alpha \gamma y} (1 - e^{-\alpha y})^{\delta}, \ y > 0,$$

which is the density of the BE distribution introduced by Nadarajah and Kotz (2006).

4 Expansions for the Distribution and Density Functions

We now give simple expansions for the cdf of the GKw distribution. If |z| < 1 and $\delta > 0$ is a non-integer real number, we have

$$(1-z)^{\delta} = \sum_{j=0}^{\infty} (-1)^{j} (\delta)_{j} z^{j}, \tag{11}$$

where $(\delta)_j = \delta(\delta - 1) \dots (\delta - j + 1)$ (for $j = 0, 1, \dots$) is the descending factorial. Clearly, if δ is a positive integer, the series stops at $j = \delta$. Using the series expansion (11) and the representation for the GKw cdf (6), we obtain

$$F(x; \boldsymbol{\theta}) = \int_0^{G_1(x; \alpha, \beta)} \frac{\lambda}{B(\gamma, \delta + 1)} y^{\gamma \lambda - 1} \sum_{j=0}^{\infty} (\delta)_j (-1)^j y^{\lambda j} dy$$

if δ is a non-integer real number. By simple integration, we have

$$F(x; \boldsymbol{\theta}) = \sum_{j=0}^{\infty} \omega_j [G_1(x; \alpha, \beta)]^{\lambda(\gamma+j)}, \tag{12}$$

where

$$\omega_j = \frac{(-1)^j (\delta)_j}{(\gamma + j)B(\gamma, \delta + 1)},\tag{13}$$

and $G_1(x;\alpha,\beta)$ is given by (3). If δ is a positive integer, the sum stops at $j=\delta$.

The moments of the GKw distribution do not have closed form. In order to obtain expansions for these moments, it is convenient to develop expansions for its density function. From (12), we can write

$$f(x; \boldsymbol{\theta}) = \sum_{j=0}^{\infty} \omega_j \lambda(\gamma + j) g_1(x; \alpha, \beta) [G_1(x; \alpha, \beta)]^{\lambda(\gamma + j) - 1}.$$

If we replace $G_1(x;\alpha,\beta)$ by (3) and use (4), we obtain

$$f(x;\boldsymbol{\theta}) = \sum_{k=0}^{\infty} p_k g_1(x;\alpha,(k+1)\beta), \tag{14}$$

where $p_k = \sum_{j=0}^{\infty} \omega_j t_{j,k}$, with $t_{j,k} = (\phi)_k \lambda(\gamma + j)(-1)^k/(k+1)$. Here, $\phi = (\gamma + j)\lambda - 1$ and $g_1(x; \alpha, (k+1)\beta)$ denotes the Kw $(\alpha, (k+1)\beta)$ density function with parameters α and $(k+1)\beta$. Further, we can express (14) as a mixture of power densities, since the Kw density (4) can also be written as a mixture of power densities. After some algebra, we obtain

$$f(x;\boldsymbol{\theta}) = \sum_{i=0}^{\infty} v_i \, x^{(i+1)\alpha - 1},\tag{15}$$

where

$$v_i = (-1)^i \alpha \beta \sum_{k=0}^{\infty} (k+1) ((k+1)\beta - 1)_i p_k.$$

Equations (14) and (15) are the main results of this section. They can provide some mathematical properties of the GKw distribution from the properties of the Kw and power distributions, respectively.

5 Moments and Moment Generating Function

Let X be a random variable having the GKw distribution (7). First, we obtain an infinite sum representation for the rth ordinary moment of X, say $\mu'_r = E(X^r)$. From (14), we can write

$$\mu_r' = \sum_{k=0}^{\infty} p_k \, \tau_r(k), \tag{16}$$

where $\tau_r(k) = \int_0^1 x^r g_1(x; \alpha, (k+1)\beta) dx$ is the *rth* moment of the $\text{Kw}(\alpha, (k+1)\beta)$ distribution which exists for all $r > -\alpha$. Using a result due to Jones (2009, Section 3), we have

$$\tau_r(k) = (k+1)\beta B\left(1 + \frac{r}{\alpha}, (k+1)\beta\right). \tag{17}$$

Hence, the moments of the GKw distribution follow directly from (16) and (17). The central moments (μ_s) and cumulants (κ_s) of X are easily obtained from the ordinary moments by $\mu_s = \sum_{k=0}^s \binom{s}{k} (-1)^k \mu_1'^s \mu_{s-k}'$ and $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, $\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$, $\kappa_5 = \mu_5' - 5\mu_4'\mu_1' - 10\mu_3'\mu_2' + 20\mu_3'\mu_1'^2 + 30\mu_2'^2\mu_1' - 60\mu_2'\mu_1'^3 + 24\mu_1'^5$, $\kappa_6 = \mu_6' - 6\mu_5'\mu_1' - 15\mu_4'\mu_2' + 30\mu_4'\mu_1'^2 - 10\mu_3'^2 + 120\mu_3'\mu_2' + 120\mu_3'\mu_1'^3 + 30\mu_2'^3 - 270\mu_2'^2\mu_1'^2 + 360\mu_2'\mu_1'^4 - 120\mu_1'^6$, etc., respectively. The rth descending factorial moment of X is

$$\mu'_{(r)} = E[X^{(r)}] = E[X(X-1) \times \dots \times (X-r+1)] = \sum_{m=0}^{r} s(r,m) \, \mu'_m,$$

where s(r,m) is the Stirling number of the first kind given by $s(r,m) = (m!)^{-1} d^m x^{(r)} / dx^m|_{x=0}$. It counts the number of ways to permute a list of r items into m cycles. Thus, the factorial moments of X are given by

$$\mu'_{(r)} = \sum_{k=0}^{\infty} p_k \sum_{m=0}^{r} s(r, m) \, \tau_m(k).$$

The moment generating function of the GKw distribution, say M(t), is obtained from (15) as

$$M(t) = \sum_{i=0}^{\infty} v_i \int_0^1 x^{(i+1)\alpha - 1} \exp(tx) dx.$$

By changing variable, we have

$$M(t) = \sum_{i=0}^{\infty} v_i t^{-(i+1)\alpha} \int_0^t u^{(i+1)\alpha - 1} \exp(-u) du$$

and then M(t) reduces to the linear combination

$$M(t) = \sum_{i=0}^{\infty} v_i \frac{\gamma((i+1)\alpha, t)}{t^{(i+1)\alpha}},$$

where $\gamma(a,x) = \int_0^a u^{a-1} \exp(-u) du$ denotes the incomplete gamma function.

6 Quantile Function

We can write (8) as $F(x; \boldsymbol{\theta}) = I_z(\gamma, \delta + 1) = u$, where $z = [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}$. From Wolfram's website¹ we can obtain some expansions for the inverse of the incomplete beta function, say $z = Q_B(u)$, one of which is

$$z = Q_B(u) = a_1v + a_2v^2 + a_3v^3 + a_4v^4 + O(v^{5/\gamma}),$$

where $v = [\gamma uB(\gamma, \delta + 1)]^{1/\gamma}$ for $\gamma > 0$ and $a_0 = 0$, $a_1 = 1$, $a_2 = \delta/(\gamma + 1)$,

$$a_3 = \frac{\delta[\gamma^2 + 3(\delta + 1)\gamma - \gamma + 5\delta + 1]}{2(\gamma + 1)^2(\gamma + 2)},$$

$$a_4 = \delta \{ \gamma^4 + (6\delta + 5)\gamma^3 + (\delta + 3)(8\delta + 3)\gamma^2 + [33(\delta + 1)^2 - 30\delta + 26]\gamma + (\delta + 1)(31\delta - 16) + 18 \} / [3(\gamma + 1)^3(\gamma + 2)(\gamma + 3)], \dots$$

The coefficients a_i 's for $i \geq 2$ can be derived from the cubic recursion (Steinbrecher and Shaw, 2007)

$$a_{i} = \frac{1}{[i^{2} + (\gamma - 2)i + (1 - \gamma)]} \left\{ (1 - \rho_{i,2}) \sum_{r=2}^{i-1} a_{r} a_{i+1-r} [r(1 - \gamma)(i - r) - r(r - 1)] + \sum_{r=1}^{i-1} \sum_{s=1}^{i-r} a_{r} a_{s} a_{i+1-r-s} [r(r - \gamma) + s(\gamma + \beta - 2) + (i + 1 - r - s)] \right\},$$

where $\rho_{i,2}=1$ if i=2 and $\rho_{i,2}=0$ if $i\neq 2$. In the last equation, we note that the quadratic term only contributes for $i\geq 3$. Hence, the quantile function $Q_{GKw}(u)$ of the GKw distribution can be written as $Q_{GKw}(u)=\{1-[1-Q_B(u)^{1/\lambda}]^{1/\beta}\}^{1/\alpha}$.

7 Mean Deviations

If X has the GKw distribution, we can derive the mean deviations about the mean $\mu'_1 = E(X)$ and about the median M from

$$\delta_1 = \int_0^1 |x - \mu_1'| f(x; \boldsymbol{\theta}) dx$$
 and $\delta_2 = \int_0^1 |x - M| f(x; \boldsymbol{\theta}) dx$,

¹http://functions.wolfram.com/06.23.06.0004.01

respectively. From (8), the median M is the solution of the nonlinear equation

$$I_{\lceil 1-(1-M^{\alpha})^{\beta}\rceil^{\lambda}}(\gamma,\delta+1)=1/2.$$

These measures can be calculated using the relationships

$$\delta_1 = 2 \left[\mu_1' F(\mu_1'; \boldsymbol{\theta}) - J(\mu_1'; \boldsymbol{\theta}) \right] \text{ and } \delta_2 = \mu_1' - 2J(M; \boldsymbol{\theta}).$$

Here, the integral $J(a; \boldsymbol{\theta}) = \int_0^a x f(x; \boldsymbol{\theta}) dx$ is easily calculated from the density expansion (15) as

$$J(a; \boldsymbol{\theta}) = \sum_{i=0}^{\infty} \frac{v_i a^{(i+1)\alpha+1}}{(i+1)\alpha+1}.$$

We can use this result to obtain the Bonferroni and Lorenz curves. These curves have applications not only in economics to study income and poverty, but also in other fields, such as reliability, demography, insurance and medicine. They are defined by

$$B(p; \boldsymbol{\theta}) = \frac{J(q; \boldsymbol{\theta})}{p\mu'_1}$$
 and $L(p; \boldsymbol{\theta}) = \frac{J(q; \boldsymbol{\theta})}{\mu'_1}$,

respectively, where $q = F^{-1}(p; \boldsymbol{\theta})$.

8 Moments of Order Statistics

The density function of the *ith* order statistic $X_{i:n}$, say $f_{i:n}(x; \boldsymbol{\theta})$, in a random sample of size n from the GKw distribution, is given by (for $i = 1, \dots, n$)

$$f_{i:n}(x; \boldsymbol{\theta}) = \frac{1}{B(i, n - i + 1)} f(x; \boldsymbol{\theta}) F(x; \boldsymbol{\theta})^{i-1} \{1 - F(x; \boldsymbol{\theta})\}^{n-1}, \ 0 < x < 1.$$
 (18)

The binomial expansion yields

$$f_{i:n}(x;\boldsymbol{\theta}) = \frac{1}{B(i,n-i+1)} f(x;\boldsymbol{\theta}) \sum_{j=0}^{n-1} {n-1 \choose j} (-1)^j F(x;\boldsymbol{\theta})^{i+j-1},$$

and using and integrating (15) we arrive at

$$f_{i:n}(x;\boldsymbol{\theta}) = \frac{1}{B(i,n-i+1)} \left(\sum_{t=0}^{\infty} v_t \, x^{(t+1)\alpha-1} \right) \sum_{j=0}^{n-1} {n-1 \choose j} (-1)^j \left(\sum_{s=0}^{\infty} v_s^{\star} \, x^{(s+1)\alpha} \right)^{i+j-1},$$

where $v_s^* = v_s[(s+1)\alpha]^{-1}$.

We use the following expansion for a power series raised to a integer power (Gradshteyn and Ryzhik, 2000, Section 0.314)

$$\left(\sum_{j=0}^{\infty} a_j x^j\right)^p = \sum_{j=0}^{\infty} c_{j,p} x^j,\tag{19}$$

where p is any positive integer number, $c_{0,p} = a_0^p$ and $c_{s,p} = (sa_0)^{-1} \sum_{j=1}^s (jp-s+j)a_jc_{s-j,p}$ for all $s \ge 1$.] We can write

$$f_{i:n}(x;\boldsymbol{\theta}) = \frac{1}{B(i,n-i+1)} \sum_{j=0}^{n-1} {n-1 \choose j} (-1)^j \sum_{s,t=0}^{\infty} v_t \, e_{s,i+j-1} \, x^{(s+t+i+j)\alpha-1},$$

where $e_{0,i+j-1} = v_0^{\star (i+j-1)}$ and (for $s \geq 1)$

$$e_{s,i+j-1} = (sv_0^*)^{-1} \sum_{m=1}^s [m(i+j-1) - s + m] v_m^* e_{s-m,i+j-1}.$$

The rth moment of the ith order statistic becomes

$$E(X_{i:n}^r) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-1} {n-1 \choose j} (-1)^j \sum_{s,t=0}^{\infty} \frac{v_t \, e_{s,i+j-1}}{(r+s+t+i+j)\alpha}. \tag{20}$$

We now obtain another closed form expression for the moments of the GKw order statistics using a general result due to Barakat and Abdelkader (2004) applied to the independent and identically distributed case. For a distribution with pdf $f(x; \theta)$ and cdf $F(x; \theta)$, we can write

$$E(X_{i:n}^r) = r \sum_{m=r-i+1}^{n} (-1)^{m-n+i-1} \binom{m-1}{n-i} \binom{n}{m} I_m(r),$$

where

$$I_m(r) = \int_0^1 x^{r-1} \{1 - F(x; \boldsymbol{\theta})\}^m dx.$$

For a positive integer m, we have

$$I_m(r) = \int_0^1 x^{r-1} \sum_{p=0}^m {m \choose p} (-1)^p [F(x; \boldsymbol{\theta})]^p dx.$$

By replacing (12) in the above equation we have

$$I_m(r) = \sum_{p=0}^m (-1)^p \binom{m}{p} \int_0^1 x^{r-1} \left(\sum_{j=0}^\infty \omega_j [G_1(x;\alpha,\beta)]^{\lambda(\gamma+j)} \right)^p dx. \tag{21}$$

Equations (19) and (21) yield

$$I_m(r) = \sum_{p=0}^{m} {m \choose p} (-1)^p \int_0^1 x^{r-1} \sum_{j=0}^{\infty} c_{j,p} [G_1(x; \alpha, \beta)]^{\lambda(\gamma+j)} dx.$$

By replacing $G_1(x; \alpha, \beta)$ by (3) and using (11) we obtain

$$I_m(r) = \sum_{p=0}^m {m \choose p} (-1)^p \sum_{j,w=0}^\infty (-1)^w c_{j,p} (\psi)_w \int_0^1 x^{r-1} (1-x^\alpha)^{w\beta} dx,$$

where $\psi = \lambda(\gamma + j)$. Since $B(a/b,c) = b \int_0^1 w^{a-1} (1 - w^b)^{c-1} dw$ for a,b,c > 0 (Gupta and Nadarajah, 2004b), we have

$$I_m(r) = \sum_{p=0}^{m} \sum_{j,w=0}^{\infty} s_{p,j,w} B(\frac{r}{\alpha}, \beta w + 1),$$

where

$$s_{p,j,w} = \frac{(-1)^{p+w}m!}{\alpha(m-p)!p!}c_{j,p}(\psi)_w.$$

Finally, $E(X_{i\cdot n}^r)$ reduces to

$$E(X_{i:n}^r) = r \sum_{m=n-i+1}^{n} \left\{ (-1)^{m-n+i-1} \binom{m-1}{n-i} \binom{n}{m} \sum_{p=0}^{m} \sum_{j,w=0}^{\infty} s_{p,j,w} B\left(\frac{r}{\alpha}, \beta w + 1\right) \right\}.$$
 (22)

Equations (20) and (22) are the main results of this section. The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They are linear functions of expected order statistics defined by (Hoskings, 1990)

$$\lambda_{r+1} = (r+1)^{-1} \sum_{k=0}^{r} (-1)^k {r \choose k} E(X_{r+1-k:r+1}), \ r = 0, 1, \dots$$

The first four L-moments are $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. These moments have several advantages over the ordinary moments. For example, they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. From (22) applied for the means (r = 1), we can obtain expansions for the L-moments of the GKw distribution.

9 Rényi Entropy

The entropy of a random variable X with density function f(x) is a measure of variation of the uncertainty. One of the popular entropy measures is the Rényi entropy given by

$$\mathcal{J}_R(\rho) = \frac{1}{1-\rho} \log \left[\int f^{\rho}(x) dx \right], \ \rho > 0, \ \rho \neq 1.$$
 (23)

From (15), we have

$$f(x; \boldsymbol{\theta})^{\rho} = \left(\sum_{i=0}^{\infty} v_i x^{(i+1)\alpha-1}\right)^{\rho}.$$

In order to obtain an expansion for the above power series for $\rho > 0$, we can write

$$f(x;\boldsymbol{\theta})^{\rho} = \sum_{j=0}^{\infty} {\rho \choose j} (-1)^{j} \left\{ 1 - \left(\sum_{i=0}^{\infty} v_{i} x^{(i+1)\alpha - 1} \right) \right\}^{j}$$
$$= \sum_{j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} {\rho \choose j} {j \choose r} x^{(\alpha - 1)r} \left(\sum_{i=0}^{\infty} v_{i} x^{i\alpha} \right)^{r}.$$

Using equation (19), we obtain

$$f(x;\boldsymbol{\theta})^{\rho} = \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} \binom{\rho}{j} \binom{j}{r} d_{i,r} x^{(i+r)\alpha-r},$$

where $d_{0,r} = v_0^r$ and $d_{s,r} = (sv_0)^{-1} \sum_{m=1}^s (mr - s + m) v_m d_{s-m,r}$ for all $s \ge 1$. Hence,

$$\mathcal{J}_R(\rho) = \frac{1}{1-\rho} \log \left[\sum_{i,j=0}^{\infty} \sum_{r=0}^{j} \frac{(-1)^{j+r} \binom{\rho}{j} \binom{j}{r} d_{i,r}}{(i+r)\alpha - r + 1} \right].$$

10 Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from the $GKw(\lambda, \alpha, \beta, \gamma, \delta)$ distribution. From (7) the log-likelihood function is easy to derive. It is given by

$$\ell(\boldsymbol{\theta}) = n \log(\lambda) + n \log(\alpha) + n \log(\beta) - n \log[B(\gamma, \delta + 1)] + (\alpha - 1) \sum_{i=1}^{n} \log(x_i) + (\beta - 1) \sum_{i=1}^{n} \log(1 - x_i^{\alpha}) + (\gamma \lambda - 1) \sum_{i=1}^{n} \log[1 - (1 - x_i^{\alpha})^{\beta}] + \delta \sum_{i=1}^{n} \log[1 - \{1 - (1 - x_i^{\alpha})^{\beta}\}^{\lambda}].$$

By taking the partial derivatives of the log-likelihood function with respect to λ , α , β , γ and δ , we obtain the components of the score vector, $U(\boldsymbol{\theta}) = (U_{\alpha}, U_{\beta}, U_{\gamma}, U_{\delta}, U_{\lambda})$. They are given by

$$U_{\alpha}(\boldsymbol{\theta}) = \frac{n}{\alpha} + \sum_{i=1}^{n} [1 - (\beta - 1)z_{i}] \log(x_{i}) + (\gamma \lambda - 1) \sum_{i=1}^{n} \frac{\dot{y}_{i(\alpha)}}{y_{i}} - \delta \lambda \sum_{i=1}^{n} v_{i} \dot{y}_{i(\alpha)},$$

$$U_{\beta}(\boldsymbol{\theta}) = \frac{n}{\beta} + \sum_{i=1}^{n} \log(1 - x_{i}^{\alpha}) + (\gamma \lambda - 1) \sum_{i=1}^{n} \frac{\dot{y}_{i(\beta)}}{y_{i}} - \lambda \delta \sum_{i=1}^{n} v_{i} \dot{y}_{i(\beta)},$$

$$U_{\gamma}(\boldsymbol{\theta}) = -n[\psi(\gamma) - \psi(\gamma + \delta + 1)] + \lambda \sum_{i=1}^{n} \log(y_{i}),$$

$$U_{\delta}(\boldsymbol{\theta}) = -n[\psi(\delta + 1) - \psi(\gamma + \delta + 1)] + \sum_{i=1}^{n} \log(1 - y_{i}^{\lambda}),$$

$$U_{\lambda}(\boldsymbol{\theta}) = \frac{n}{\lambda} + \sum_{i=1}^{n} [\gamma - \delta y_{i} v_{i}] \log(y_{i}),$$

where $\psi(\cdot)$ is the digamma function, $y_i = 1 - (1 - x_i^{\alpha})^{\beta}$, $v_i = y_i^{\lambda - 1} (1 - y_i^{-\lambda})^{-1}$, $z_i = x_i^{\alpha} (1 - x_i^{\alpha})^{-1}$, $\dot{y}_{i(\alpha)} = \partial y_i / \partial \alpha = -\beta x_i^{\alpha} (1 - x_i^{\alpha})^{\beta - 1} \log(x_i)$ and $\dot{y}_{i(\beta)} = \partial y_i / \partial \beta = -(1 - x_i^{\alpha})^{\beta} \log(1 - x_i^{\alpha})$. For interval estimation and hypothesis tests on the model parameters, the observed information matrix is required. The observed information matrix $J = J(\theta)$ is given in the Appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space, the approximate distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is multivariate normal $N_5(\mathbf{0}, I^{-1}(\boldsymbol{\theta}))$, where $\widehat{\boldsymbol{\theta}}$ is the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$ and $I(\boldsymbol{\theta})$ is the expected information matrix. This approximation is also valid if $I(\boldsymbol{\theta})$ is replaced by $J(\widehat{\boldsymbol{\theta}})$.

The multivariate normal $N_5(\mathbf{0}, J^{-1}(\widehat{\boldsymbol{\theta}}))$ distribution can be used to construct approximate confidence regions. The well-known likelihood ratio (LR) statistic can be used for testing hypotheses on the model parameters in the usual way. In particular, this statistic is useful to check if the fit using the GKw distribution is statistically superior to a fit using the BKw, EKw and EKw distributions for a given data set. For example, the test of $H_0: \lambda = 1$ versus $H_1: \lambda \neq 1$ is equivalent to compare the BKw distribution with the GKw distribution and the LR statistic reduces to $W = 2[\ell(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \widehat{\lambda}) - \ell(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}, 1)]$, where $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\theta}}$ are the unrestricted and restricted MLEs of $\boldsymbol{\theta}$, respectively. Under the null hypothesis, w is asymptotically distributed as χ_1^2 . For a given level ζ , the LR test rejects H_0 if w exceeds the $(1-\zeta)$ -quantile of the χ_1^2 distribution.

11 Application

This section contains an application of the GKw distribution to real data. The data are the observed percentage of children living in households with per capita income less than R\$ 75.50 in 1991 in 5509 Brazilian municipal districts. The data were extracted from the Atlas of

Brazil Human Development database available at http://www.pnud.org.br/. The histogram of the data is shown in Figure 2 along with the estimated densities of the GKw distribution and some special sub-models. Apparently, the GKw distribution gives the best fit.

The GKw model includes some sub-models described in Section 3 as especial cases and thus allows their evaluation relative to each other and to a more general model. As mentioned before, we can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain the LR statistics for testing some sub-models of the GKw distribution. We test $H_0: (\alpha, \beta, \lambda) = (1, 1, 1)$ versus $H_1: H_0$ is not true, i.e. we compare the GKw model with the beta model. In this case, $w = 2\{1510.7 - 1271.6)\} = 239.1$ (p-value< 0.001) indicates that the GKw model gives a better representation of the data than the beta distribution. Further, the LR statistic for testing $H_0: \lambda = 1$ versus $H_1: \lambda \neq 1$, i.e. to compare the GKw model with the BKw model, is w = 2(1510.7 - 1383.6) = 254.2 (p-value< 0.001). It also yields favorable indication for the GKw model. Table 1 lists the MLEs of the model parameters (standard errors in parentheses) for different models. The computations were carried out using the subroutine MAXBFGS implemented in the 0x matrix programming language (Doornik, 2007).

 $\ell(\boldsymbol{\theta})$ Distribution β δ λ α γ 15.7803 1510.6670 GKw18.1161 1.81320.73030.0609 (0.1829)(0.0219)(0.0057)(0.0008)(0.8908)BKw0.02470.184926.0933 17.3768 1383.5690(0.1054)(0.0003)(0.0005)(0.0739)KKw2.71910.09680.465479.99991405.0650(0.0086)(0.0060)(0.0005)(1.0915)PKw17.9676 0.16471.1533 1237.5800(0.2421)(0.0019)(0.0054)BP0.159016.73130.29411269.9760 (0.0018)(0.1998)(0.0129)

1278.7860

1271.5610

Table 1: MLEs of the model parameters.

12 Conclusions

Kw

Beta

2.4877

(0.0295)

1.3369

(0.0180)

We introduce a new five-parameter continuous distribution on the standard unit interval which generalizes the beta, Kumaraswamy (Kumaraswamy, 1980) and McDonald (McDonald,

2.5678

(0.0317)

0.3010

(0.0147)

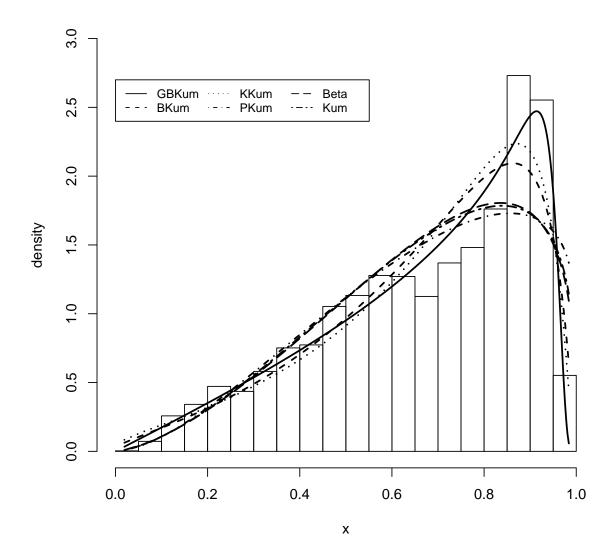


Figure 2: Histogram and estimated pdf's for the percentage of children living in households with per capita income less than R\$ 75.50 (1991) in 5509 Brazilian municipal districts.

1984) distributions and includes as special sub-models other distributions discussed in the literature. We refer to the new model as the generalized Kumaraswamy distribution and study some of its mathematical properties. We demonstrate that the generalized Kumaraswamy density function can be expressed as a mixture of Kumaraswamy and power densities. We provide the moments and a closed form expression for the moment generating function. Explicit expressions are derived for the mean deviations, Bonferroni and Lorenz curves and Rényi's entropy. The density of the order statistics can also be expressed in terms of an infinite mixture of power densities. We obtain two explicit expressions for their moments. Parameter estimation is approached by maximum likelihood. The usefulness of the new distribution is illustrated in an analysis of real data. We hope that the proposed extended model may attract wider applications in the analysis of proportions data.

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Appendix

The elements of the observed information matrix $J(\theta)$ for $(\alpha, \beta, \gamma, \delta, \lambda)$ are

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} - (\beta - 1) \sum_{i=1}^n \dot{z}_{i(\alpha)} \log(x_i) + (\gamma \lambda - 1) \sum_{i=1}^n \left\{ \frac{\ddot{y}_{i(\alpha)}}{y_i} - \left(\frac{\dot{y}_{i(\alpha)}}{y_i} \right)^2 \right\} - \delta \lambda \sum_{i=1}^n (\dot{v}_{i(\alpha)} \dot{y}_{i(\alpha)} + v_i \ddot{y}_{i(\alpha)}),$$

$$J_{\alpha\beta} = -\sum_{i=1}^n z_i \log(x_i) + (\gamma \lambda - 1) \sum_{i=1}^n \left\{ \frac{\ddot{y}_{i(\alpha\beta)}}{y_i} - \frac{\dot{y}_{i(\alpha)} \dot{y}_{i(\beta)}}{y_i^2} \right\} - \delta \lambda \sum_{i=1}^n (\dot{v}_{i(\beta)} \dot{y}_{i(\alpha)} + v_i \ddot{y}_{i(\alpha\beta)}),$$

$$J_{\alpha\gamma} = \lambda \sum_{i=1}^n \frac{\dot{y}_{i(\alpha)}}{y_i}, \quad J_{\alpha\delta} = -\lambda \sum_{i=1}^n v_i \dot{y}_{i(\alpha)}, \quad J_{\alpha\lambda} = \sum_{i=1}^n \{\gamma/y_i - \delta v_i\} \dot{y}_{i(\alpha)},$$

$$J_{\beta\beta} = -\frac{n}{\beta^2} + (\gamma\lambda - 1) \sum_{i=1}^n \left\{ \frac{\ddot{y}_{i(\beta)}}{y_i} - \left(\frac{\dot{y}_{i(\beta)}}{y_i} \right)^2 \right\} - \delta\lambda \sum_{i=1}^n (\dot{v}_{i(\beta)} \dot{y}_{i(\beta)} + v_i \ddot{y}_{i(\beta)}),$$

$$J_{\beta\gamma} = \lambda \sum_{i=1}^n \frac{\dot{y}_{i(\beta)}}{y_i}, \quad J_{\beta\delta} = -\lambda \sum_{i=1}^n v_i \dot{y}_{i(\beta)}, \quad J_{\beta\lambda} = \gamma \sum_{i=1}^n \frac{\dot{y}_{i(\beta)}}{y_i} - \delta \sum_{i=1}^n v_i \dot{y}_{i(\beta)},$$

$$J_{\gamma\gamma} = -n\{\psi'(\gamma) - \psi'(\gamma + \delta + 1)\}, \quad J_{\gamma\delta} = n\psi'(\gamma + \delta + 1), \quad J_{\gamma\lambda} = \sum_{i=1}^n \log(y_i),$$

$$J_{\delta\delta} = -n\{\psi'(\delta + 1) - \psi'(\gamma + \delta + 1)\}, \quad J_{\delta\lambda} = -\sum_{i=1}^n y_i v_i \log(y_i),$$

$$J_{\lambda\lambda} = -\frac{n}{\lambda^2} - \delta \sum_{i=1}^n y_i \dot{v}_{i(\lambda)} \log(y_i),$$

where $\dot{z}_{i(\alpha)} = \partial z_i/\partial \alpha = (1+z_i)z_i \log(x_i)$, $\ddot{y}_{i(\alpha)} = \partial^2 y_i/\partial \alpha^2 = \{1-(\beta-1)z_i\}\dot{y}_{i(\alpha)} \log(x_i)$, $\ddot{y}_{i(\beta)} = \partial^2 y_i/\partial \beta^2 = \dot{y}_{i(\beta)} \log(1-x_i^{\alpha})$, $\ddot{y}_{i(\alpha\beta)} = \partial^2 y_i/\partial \alpha \partial \beta = \{1/\beta + \log(1+x_i^{\alpha})\}\dot{y}_{i(\alpha)}$, $\dot{v}_{i(\alpha)} = \partial v_i/\partial \alpha = \{(\lambda-1)/y_i + \lambda v_i\}v_i\dot{y}_{i(\alpha)}$, $\dot{v}_{i(\beta)} = \partial v_i/\partial \beta = \{(\lambda-1)/y_i + \lambda v_i\}v_i\dot{y}_{i(\beta)}$, $\dot{v}_{i(\lambda)} = \partial v_i/\partial \lambda = (1+y_iv_i)v_i \log(y_i)$ and $\psi'(\cdot)$ is the first derivative of the digamma function.

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